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VARIATIONAL FORMULATION OF HYPERBOLIC CONSERVATION LAWS

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In memory of Peter Lax

Abstract. Entropy functions played a key role in the development of mathematical theory for hyperbolic conservation laws. The notion of entropy, which is intimately connected with symmetry, is an extension *imposed* on nonlinear systems of conservation laws. In this context, Friedrichs raised the question whether the assumed symmetries can also be derived. We introduce a variational formulation that addresses Friedrichs' question: an entropy function is derived as a stationary object, from which we deduce—rather than impose—the maximum entropy production principle of Dafermos.

"There is no theory for the initial value problem for compressible flows in two space dimensions once shocks show up, much less in three space dimensions. This is a scientific scandal and a challenge." P. D. Lax [61]

Contents

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Dedication. Peter Lax was one of the preeminent mathematicians of the second half of the twentieth century and the leading ambassador for modern applied mathematics [59]. He served as a role model for the generations of mathematicians who followed him, myself included.

To grasp the scale of his mathematical legacy, one can recall the concepts that bear his signature: the Lax equivalence theorem, the Lax-Milgram lemma, the Hopf-Lax formula, the Lax-Friedrichs and Lax-Wendroff methods, the Glimm-Lax memoir, Lax-Levermore theory, Lax pairs, Lax-Phillips scattering theory, the Lax entropic shock conditions, the HLL Riemann solver, not to mention his pioneering works on the Riemann problem in hyperbolic conservation laws, and on Fourier integral operators. A deeper look into Lax's contributions can be found in [60].

I first met Peter during a trip to Northern California in the summer of 1981. I was a young postdoc from Caltech, giving a seminar at Stanford. When I mentioned that I would be visiting New York later that summer, he offered, entirely on the spot: "Why don't you stay in my New York apartment?" That "you" included our daughter, who was three years old at the time. It remains one of those unforgettable life events that exemplified Peter's generosity. When I later shared this memory with his son, Dr. Jim Lax, he replied, "My parents were like that. My friends referred to their apartment as Hotel Lax."
This was quintessentially Lax. He was not merely an iconic mathematical figure for my generation; he was an extraordinary human being. He was an admirer of John von Neumann [62] and, as he himself noted, not the first Hungarian-born to serve as President of the American Mathematical Society. Both were part of the Hungarian group of scientists affectionately referred to as "The Martians" by their peers [82].

Peter had style and class, both in his life and in his mathematics. While his impact cannot be adequately captured in a few words, his legacy will undoubtedly endure.

1. Introduction — a question of Friedrichs about symmetry.

In 1979, Friedrichs concluded his von Neumann lecture with the following question [28]:

"*For the systems of equations I have discussed here, the symmetry feature was a derived property. Now, in many branches of physics ... symmetries play a fundamental role, but all these symmetries — as it seems to me — are assumed and not derived. I now wonder whether or not ... symmetries can also be derived from the overdeterminancy of basic conservation equations?*"

This paper describes an attempt to answer Friedrichs' question in the context of hyperbolic systems of conservation laws[1]

$$\mathbf{u}_t + \sum_{j=1}^{d} \mathbf{f}_j(\mathbf{u})_{x_j} = 0, \qquad \mathbf{u}(t,\cdot) : \mathbb{R}^d_{\mathbf{x}} \mapsto \mathbb{R}^N, \quad t \geqslant 0,\ \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d. \quad (1)$$

Here, $\mathbf{u} = (u^1, \ldots, u^N)^\top$ is an N-vector of conserved quantities in $L^1_{loc} \cap L^\infty(\mathbb{R}^d)$, which are balanced by flux vectors $\mathbf{f}_j(\mathbf{u}) = (f^1_j(\mathbf{u}), \ldots, f^N_j(\mathbf{u}))^\top : \mathbb{R}^N \mapsto \mathbb{R}^N$.

[1] Dafermos concludes his sketch of the early history of hyperbolic conservation laws, writing [16, Introduction], "The next major milestone ... is the landmark paper by Lax [55], which coins the term "hyperbolic conservation law" and launches the field as a new principal branch in the theory of partial differential equations. This was accomplished by distilling, generalizing and formalizing the raw material that had accumulated over the years ... It is fair to say that Lax's paper set the direction for the development of the field of hyperbolic conservation laws over the past fifty years."

Integrating (1) yields the conservative weak form for the N components of the equations, $u_t^\alpha + \operatorname{div}\mathbf{f}^\alpha(\mathbf{u}) = 0$,

$$\int_\Omega u^\alpha(t,\cdot)\,\mathrm{d}\mathbf{x}\Big|_{t=t_1}^{t=t_2} + \int_{t=t_1}^{t_2}\int_{\partial\Omega} \mathbf{f}^\alpha\cdot\mathbf{n}\,\mathrm{d}S\,\mathrm{d}t = 0, \qquad \mathbf{f}^\alpha(\mathbf{u}) = \big(f_1^\alpha(\mathbf{u}),\ldots,f_d^\alpha(\mathbf{u})\big),\quad \alpha=1,\ldots,N. \tag{2}$$

It states that the change over time in the amount of matter inside an arbitrary spatial domain $\Omega\subset\mathbb{R}^d$ is solely due to the flux of that matter across the boundary of Ω. We recall that (2) is equivalent to the usual notion of weak solutions, where (1) is interpreted in the distributional sense, [57][2].

Differentiation of (1) puts it in the form of a quasi-linear system

$$\mathbf{u}_t + \sum_{j=1}^d A_j(\mathbf{u})\mathbf{u}_{x_j} = 0, \qquad A_j(\mathbf{u}) := \frac{\partial\mathbf{f}_j(\mathbf{u})}{\partial\mathbf{u}}. \tag{3}$$

The symmetry mentioned in Friedrichs' question is connected with the notion of an entropy. A nonlinear scalar function, $\eta=\eta(\mathbf{u})$, is an entropy function associated with (1), if there exists an entropy flux, $\mathbf{q}^\eta(\mathbf{u}) = (q_1(\mathbf{u}),\ldots,q_d(\mathbf{u}))^\top$, such that[3]

$$\nabla\eta(\mathbf{u})^\top A_j(\mathbf{u}) = \nabla q_j(\mathbf{u})^\top, \qquad j=1,\ldots,d. \tag{4}$$

It follows that the conservation law (1) admits an extension in terms of the entropy pair $(\eta,\mathbf{q}^\eta)$:

$$\Big\langle\nabla\eta(\mathbf{u}),\mathbf{u}_t + \sum_{j=1}^d \mathbf{f}_j(\mathbf{u})_{x_j}\Big\rangle = \eta(\mathbf{u})_t + \sum_j \nabla\eta(\mathbf{u})^\top A_j(\mathbf{u})\mathbf{u}_{x_j} = \eta(\mathbf{u})_t + \nabla_{\mathbf{x}}\cdot\mathbf{q}^\eta(\mathbf{u}).$$

Of course, such an extension holds for any linear combination of the components of $\mathbf{u}$; the essence, therefore, is imposing the compatibility relation (4) for nonlinear entropy functions. Let $\mathcal{C}(\mathbb{R}^N,\mathbb{R})$ denote the cone of strictly convex functions, then one is led to the requirement that for all $\eta\in\mathcal{C}$

$$\eta(\mathbf{u})_t + \nabla_{\mathbf{x}}\cdot\mathbf{q}^\eta(\mathbf{u}) \leqslant 0. \tag{5}$$

This is the celebrated *entropy inequality*, imposed as a selection criterion of physically relevant weak solutions for (1) [56, 52].

In [29], Friedrichs and Lax observed that the entropy compatibility requirement (4) is equivalent to symmetry. This follows from differentiation of (4) (here and below $D^2\zeta(\mathbf{u})$ denotes the Hessian $(D^2\zeta)_{\alpha,\beta} = \dfrac{\partial^2\zeta}{\partial u^\alpha\partial u^\beta}$),

$$(D^2\eta)A_j + T_j = D^2 q_j, \qquad \big(T_j\big)_{\alpha\beta} := \Big\langle\nabla\eta, \frac{\partial^2\mathbf{f}_j}{\partial u^\alpha\partial u^\beta}\Big\rangle.$$

Hence, since the Hessians D^2q_j and the tensors T_j are symmetric, the existence of an entropy implies that its Hessian symmetrizes the Jacobians

$$D^2\eta(\mathbf{u})A_j(\mathbf{u}) = \big(A_j(\mathbf{u})D^2\eta(\mathbf{u})\big)^\top \equiv A_j^\top(\mathbf{u})D^2\eta(\mathbf{u}), \quad j=1,\ldots,d. \tag{6}$$

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[2] Weak solutions are required to be at least integrable, and in order to make sense of the action of general nonlinear fluxes given that $\mathbf{f}(\mathbf{u})$ might not be locally integrable, also uniformly bounded, $\mathbf{u}\in L^1_{loc}\cap L^\infty(\mathbb{R}^d)$. The question whether "generalized" solutions satisfy (2) in a meaningful sense when $\mathbf{u}$ is merely in $C([0,\infty);L^1(\mathbb{R}^d))$ was addressed in [73].

[3] $\nabla\zeta(\mathbf{u})$ is the *column* gradient vector $(\zeta_{u^1}(\mathbf{u}),\ldots,\zeta_{u^N}(\mathbf{u}))$ and $\nabla\zeta(\mathbf{u})^\top$ is the *row* gradient vector.

Conversely, if the conservative system (3) is symmetrizable by a positive Hessian, $(D^2\eta)A_j = A_j^\top(D^2\eta)$, then there exists entropy flux $\mathbf{q}^\eta = (q_1,\ldots,q_d)^\top$ such that (4) holds and $(\eta,\mathbf{q}^\eta)$ forms a (convex) entropy pair.

We conclude this Introduction with two comments which elaborate on the interplay between symmetry and entropy.

1.1. **Imposing the existence of entropy symmetrizer $N \geqslant 3$.** The symmetry condition (6) clearly holds in the scalar case: When $N=1$, every convex $\eta(u)$ serves as a convex entropy. In particular, Kružkov's one-parameter family of entropy functions depending on a "dual" scalar variable c (or equivalently, the kinetic velocity variable in the kinetic formulation of [64, Corollary 1]),

$$\eta_c(u) = |u-c|, \qquad c \in \mathbb{R}, \tag{7}$$

is the main tool for development of existence, uniqueness and L^1-stability of scalar conservation laws, [52] (see also the earlier work [81, §16]). In the case of $N=2$ equations, the symmetry (6) amounts to a second-order linear equation for $D^2\eta = D^2\eta(u^1,u^2)$. In this context we mention the family of Lax entropy pairs [56], and special classes of 2×2 systems with entropy functions that arise as solutions for Euler-Poisson-Darboux equation, [19],[64, §1.1],[4], [5, §3], or Goursat problem, [72, §9.3], [16, Chap. XII]. For $N \geqslant 3$, the symmetry condition forms an over-determined system for the $N(N-1)/2$ entries of $D^2\eta$. The 3×3 system

$$\mathbf{u}_t + \begin{bmatrix} u^2 & & \\ & u^3 & \\ & & u^1 \end{bmatrix}\mathbf{u}_x = 0, \tag{8}$$

is an example for the class of "completely non-conservative" systems studied by Rozhdestvenskii, [70] (translated in [71, §7]). It is in this sense that one needs to impose the entropy symmetrizer condition (6), at least for systems with $N \geqslant 3$ equations.

1.2. **Symmetry and conservation.** The entropy Hessian in (6), $A_0^S(\mathbf{u}) := D^2\eta(\mathbf{u}) > 0$, puts the system (3) into Friedrichs' symmetric form, [27],

$$A_0^S(\mathbf{u})\mathbf{u}_t + \sum_{j=1}^d A_j^S(\mathbf{u})\mathbf{u}_{x_j} = 0, \qquad A_j^S(\mathbf{u}) := A_0^S(\mathbf{u})A_j(\mathbf{u}), \quad j=1,\ldots,d. \tag{9}$$

The system is symmetric in the sense that $A_j^S, j=0,1,\ldots,d$ are symmetric, and therefore hyperbolic in the sense that $\sum_{j=0}^d A_j^S\omega_j$ are diagonalizable with real eigenvalues. However, symmetrization comes at the expense of conservation: the symmetric $A_j^S(\mathbf{u})$ need not be Jacobian matrices which would enable to identify (9) as a system of conservation laws.

In his seminal 1961 paper entitled "An interesting class of quasilinear systems" [38], Godunov identified a class of quasilinear equations that can be written in a form that is both symmetric and in conservation form:

$$\big(\nabla L_0(\mathbf{v})\big)_t + \sum_{j=1}^d \big(\nabla L_j(\mathbf{v})\big)_{x_j} = 0, \tag{10}$$

with a quasi-linear symmetric form expressed in terms of the corresponding Hessians

$$(D^2_{\mathbf{v}} L_0)\mathbf{v}_t + \sum_{j=1}^{d} (D^2_{\mathbf{v}} L_j)\mathbf{v}_{x_j} = 0. \tag{11}$$

Godunov showed that the equations of compressible gas dynamics as well as other hyperbolic systems in mathematical physics admit the conservative form (10) for a proper choice of $L_j = L_j(\mathbf{v})$ and variables $\mathbf{v}$ that were worked out in each case, [38, 39, 40].

A final and decisive step, bridging the work of Godunov with that of Friedrichs and Lax, was taken in 1980 by Mock, [68][4]. Given a convex entropy function $\eta(\mathbf{u})$, one defines the *entropy variables*, $\mathbf{v} := \nabla\eta(\mathbf{u})$; by convexity one can, at least locally, consider the inverse mapping which we express as $\mathbf{u} = \mathbf{u}(\mathbf{v})$. Expressed in terms of these entropy variables, system (1) keeps its conservative form

$$\mathbf{u}(\mathbf{v})_t + \sum_{j=1}^{d} \mathbf{f}_j(\mathbf{u}(\mathbf{v}))_{x_j} = 0. \tag{12}$$

Moreover, the $\mathbf{v}$-dependent fluxes are now perfect gradients of an entropy potential, $\psi_0(\mathbf{v})$, and the corresponding flux potentials, $\psi_j(\mathbf{v})$,

$$\begin{aligned} \mathbf{u}(\mathbf{v}) &= \nabla\psi_0(\mathbf{v}), & \psi_0(\mathbf{v}) &:= \langle \mathbf{v}, \mathbf{u}(\mathbf{v})\rangle - \eta(\mathbf{u}(\mathbf{v})) \\ \mathbf{f}_j(\mathbf{u}(\mathbf{v})) &= \nabla\psi_j(\mathbf{v}), & \psi_j(\mathbf{v}) &:= \langle \mathbf{v}, \mathbf{f}_j(\mathbf{u}(\mathbf{v}))\rangle - q_j(\mathbf{u}(\mathbf{v})). \end{aligned}$$

This yields the quasi-linear form of (12) expressed in terms of the symmetric Hessians

$$\big(D^2_{\mathbf{v}}\psi_0\big)\mathbf{v}_t + \sum_{j=1}^{d} \big(D^2_{\mathbf{v}}\psi_j\big)\mathbf{v}_{x_j} = 0. \tag{13}$$

Compared with the Friedrichs-Lax symmetrization in (6), we observe that the entropy variables formulation in (12) symmetrizes the Jacobians A_j "on the right", since (9) yields

$$\frac{\partial \mathbf{f}_j(\mathbf{u}(\mathbf{v}))}{\partial \mathbf{v}} = A_j(\mathbf{u})(D^2\eta)^{-1} = \big(A_0^S\big)^{-1} A_j^S(\mathbf{u})\big(A_0^S\big)^{-1}. \tag{14}$$

This type of "symmetrization of the right" using the entropy variables was the main tool in the construction of entropy conservative/stable schemes in [76, 77, 79].

Observe that the potentials that dictate the entropy variable formulation (13), $\{\psi_j(\mathbf{v}), j = 0, \ldots, d\}$, coincide with the functionals, $\{L_j(\mathbf{v}), j = 0, \ldots, d\}$, in (10). Thus, we conclude that *Godunov's "interesting class of quasi-linear systems" encompasses* all *hyperbolic conservation laws endowed with (at least one) convex entropy extension.*

Example 1.1 (**The Euler system for compressible gas dynamics).** When $N \geqslant 3$, the existence of an entropy function is the exception rather than the rule. Nevertheless, many physically relevant systems are endowed with entropy functions. The primary example — the one that in fact motivated much of the theoretical development over the years — is the Euler system for compressible gas dynamics. It consists of $N = d + 2$ equations for the conservative vector function, $\mathbf{u} = \mathbf{u}^{\text{Euler}}$,

[4]Consult the concise summary of Lax [58] in response to Godunov [41].

which consists of density $\rho > 0$, momentum $\rho\mathsf{v} = \rho(\mathsf{v}_1, \ldots, \mathsf{v}_d) \in \mathbb{R}^d$, and total energy $E := \dfrac{1}{2}\rho|\mathsf{v}|^2 + \rho e$,

$$\mathbf{u}^{\text{Euler}} = \begin{bmatrix} \rho \\ \rho\mathsf{v} \\ E \end{bmatrix}, \quad \mathbf{f}_j^{\text{Euler}}(\mathbf{u}) = \begin{bmatrix} \rho\mathsf{v}_j \\ \rho\mathsf{v}_j\mathsf{v} + p\delta_{ij} \\ \mathsf{v}_j(E+p) \end{bmatrix}, \quad j = 1, \ldots, d, \quad p = (\gamma-1)\rho e. \tag{15}$$

It is endowed with a family of convex entropy functions of the specific entropy $S := \ln(p\rho^{-\gamma})$, [43]

$$\eta(\mathbf{u}^{\text{Euler}}) = -\rho h(S), \quad h' - \gamma h'' > 0, \ h' > 0. \tag{16}$$

In particular, therefore, it implies the well-known symmetrizability of the system of compressible Euler equations.[5] The entropy variables for (15) corresponding to $h(S) = \dfrac{\gamma+1}{\gamma-1}e^{S/(\gamma+1)}$ take the particularly simple form [79, §2]

$$\mathbf{v} = (p\rho)^{-\frac{\gamma}{\gamma+1}}(E, -\rho\mathsf{v}, \rho)^\top.$$

Euler fluxes, $\mathbf{f}_j^{\text{Euler}}(\mathbf{u})$, have the distinctive feature of being homogeneous of degree 1 and their homogeneity is preserved in their symmetric entropy-variable formulation. This is outlined in Appendix A.

Besides canonical examples of Euler equations (15) and other systems of mathematical physics, there are other classes of hyperbolic systems with non-empty set of entropy functions.

Example 1.2 (**Symmetric systems).** We consider the class of hyperbolic systems (3) with symmetric Jacobians,

$$A_j(\mathbf{u}) := \frac{\partial \mathbf{f}_j(\mathbf{u})}{\partial \mathbf{u}} = A_j^\top(\mathbf{u}), \qquad j = 1, \ldots, d.$$

These are precisely the systems with fluxes $\mathbf{f}_j(\mathbf{u})$ induced by potentials,

$$\zeta_j(\mathbf{u}) := \int^{\mathbf{u}} \mathbf{f}_j(\mathbf{w}) \cdot \mathrm{d}\mathbf{w}, \qquad j = 1, \ldots, d$$

such that $\nabla\zeta_j(\mathbf{u}) = \mathbf{f}_j(\mathbf{u})$. Then the quadratic "energy" $\eta(\mathbf{u}) = \frac{1}{2}|\mathbf{u}|^2$ is an entropy function [38] and hence $\eta_{\mathbf{c}}(\mathbf{u}) = \frac{1}{2}|\mathbf{u} - \mathbf{c}|^2$ is a family of entropy functions parameterized by $\mathbf{c} \in \mathbb{R}^N$. It follows that in the particular case of one-dimensional systems, $\zeta(\mathbf{u})$ symmetrizes one-dimensional symmetric systems, [78]

$$\langle \nabla\zeta(\mathbf{u}), \mathbf{u}_t + \mathbf{f}(\mathbf{u})_x \rangle = \zeta(\mathbf{u})_t + \Big(\frac{|\mathbf{f}(\mathbf{u})|^2}{2}\Big)_x, \qquad \zeta(\mathbf{u}) = \int^{\mathbf{u}} \mathbf{f}(\mathbf{w}) \cdot \mathrm{d}\mathbf{w}$$

Though $\zeta(\mathbf{u})$ need not be convex, one may consider the one parameter family of convex entropy functions $\eta_\lambda(\mathbf{u})$

$$\eta_\lambda(\mathbf{u}) = \frac{1}{2}|\mathbf{u}|^2 - \lambda\zeta(\mathbf{u}), \qquad \lambda A(\mathbf{u}) < \mathbb{I}.$$

[5] The spatial Jacobians are similar to — and therefore are symmetrizable into, the particularly simple form

$$A_j(\mathbf{u}) \sim \mathsf{v}_j \mathbb{I}_{N\times N} + c\left[\begin{array}{c|c|c} 0 & \mathbf{e}_j^\top & 0 \\ \hline \mathbf{e}_j & 0_{d\times d} & \\ \hline 0 & & 0 \end{array}\right], \qquad c := \sqrt{\frac{\gamma p}{\rho}}.$$

2. **A variational principle.** We established the relation between symmetry and entropy in the context of nonlinear conservation laws: the existence of an entropy function amounts to the assumption of existence of a symmetrizer, (6), or "symmetrization on the right" in an entropy variables formulation (12). This assumption of symmetry/existence of an entropy is *imposed* on N-systems of conservation laws when $N \geqslant 3$. We now reformulate the question of Friedrichs as follows:

> *Is it possible to derive the existence of entropy function(s) rather than merely assuming their existence?*

We aim to answer this question in terms of the following variational principle. It involves state vector functions, $\mathbf{u}(t,\cdot) \in \mathcal{W} \subset \mathrm{BV} \cap L^\infty$, and a set of strictly convex functions acting on these states, $\eta \in \mathcal{E} \subset \mathcal{C}(\mathbb{R}^N, \mathbb{R})$, which we refer to as *observables*. The precise nature of the states admitted into $\mathcal{W}$, and the observables admitted into $\mathcal{E}$, will be explored below. At this stage we postulate that $\mathcal{E}$ should be an affine space so that each $\eta \in \mathcal{E}$ generates the N-parameter family $\eta_{\mathbf{c}} \in \mathcal{E}$,

$$\text{if } \ \eta \in \mathcal{E} \ \text{ then } \ \eta_{\mathbf{c}}(\mathbf{u}) := \eta(\mathbf{u}) + \langle \mathbf{c}, \mathbf{u} \rangle \in \mathcal{E} \quad \text{for all } \ \mathbf{c} \in \mathbb{R}^N.$$

Definition 2.1 (**Variational solutions).** We say that $(\eta, \overline{\mathbf{u}})$ is a *variational solution* of (1) if η belongs to a non-empty set of observables $\mathcal{E}$, and $\overline{\mathbf{u}} \in \mathcal{W}$ is a minimizer of an action functional, $\mathcal{I}_{\eta_{\mathbf{c}}}(\mathbf{u})$, "observed" by $\eta_{\mathbf{c}}$, such that for arbitrary time interval $(t_1, t_2) \subset (0, \infty)$ and smoothly bounded domains $\Omega \subset \mathbb{R}^d$,

$$\mathcal{I}_\eta\big(\overline{\mathbf{u}}; (t_1,t_2)\big) \leqslant \mathcal{I}_{\eta_{\mathbf{c}}}\big(\mathbf{u}; (t_1,t_2)\big) \qquad \text{for all } \left\{ \begin{array}{ll} \mathbf{u}(t,\cdot) \in \mathcal{W}, & t \in (t_1,t_2) \\ \mathbf{u}(t,\cdot) = \overline{\mathbf{u}}(t,\cdot), & t \leqslant t_1 \end{array} \right\} \text{ and } \mathbf{c} \in \mathbb{R}^N. \tag{17}$$

The action functional, $\mathcal{I}_\eta$, is given by

$$\mathcal{I}_\eta(\mathbf{u}) = \mathcal{I}_\eta\big(\mathbf{u}; (t_1,t_2)\big) := \int\limits_{t=t_1}^{t_2} \int\limits_{\Omega} \Big\langle \nabla \eta(\mathbf{u}), \mathbf{u}_t + \sum_{j=1}^{d} \mathbf{f}_j(\mathbf{u})_{x_j} \Big\rangle_{\boldsymbol{\phi}} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t. \tag{18}$$

In particular, the corresponding action $\mathcal{I}_{\eta_{\mathbf{c}}}$ is defined by the same formula with η replaced by $\eta_{\mathbf{c}}$.

Remark 2.2. The space $BV \cap L^\infty$ is a natural candidate for a suitable class of solutions of (1) which must admit the emergence of jump discontinuities, both for multi-dimensional equations [52] and for one-dimensional systems [36, 72, 3, 16] (with proper restrictions, e.g., [47]). Consequently, the integrand of $\mathcal{I}_\eta(\mathbf{u})$ involves non-conservative products which are interpreted in the sense of [17]. Recall that given a vector function $\mathbf{b}(\mathbf{u}) : \mathbb{R}^N \mapsto \mathbb{R}^N$, the non-conservative product for BV data $\mathbf{u}$, denoted $[\mathbf{b}(\mathbf{u})^\top \mathbf{u}_x]_{\boldsymbol{\phi}}$, is a uniquely defined Borel measure which depends on a vector-valued Lipschitz path $\boldsymbol{\phi} : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N$, so that $\boldsymbol{\phi}(s) = \boldsymbol{\phi}(s; \mathbf{u}_-, \mathbf{u}_+)$ is connecting possible jump discontinuities $\mathbf{u}_- = \mathbf{u}(x_-)$ to $\mathbf{u}_+ = \mathbf{u}(x_+)$, while $s \in [0,1]$. Specifically, the non-conservative product at a regular point of discontinuity x has Dirac mass with $\boldsymbol{\phi}$-dependent amplitude[6]

$$\big[\mathbf{b}(\mathbf{u})^\top \mathbf{u}_x\big]_{\boldsymbol{\phi}}(x) = \int_0^1 \mathbf{b}(\boldsymbol{\phi}(s))^\top \, \frac{\mathrm{d}\boldsymbol{\phi}(s)}{\mathrm{d}s} \, \mathrm{d}s.$$

[6]This definition takes place in the $d = 1$-dimensional case. The generalization for $\mathbf{x} \in \mathbb{R}^d$ in $d > 1$ dimensions is implemented at regular points of jump discontinuities, so that their left and right limits, $\mathbf{u}_\pm = \mathbf{u}(\mathbf{x}_\pm)$, are well defined across surface of discontinuity; all other irregular points for BV functions have $(d-1)$-dimensional measure 0, [81, §9]. This enables one to define $\big[\mathbf{b}(\mathbf{u})^\top \mathbf{u}_{x_j}\big]_{\boldsymbol{\phi}}$, [17, §6.1].

We find it convenient to use the notation

$$\big\langle \mathbf{b}(\mathbf{u}), \mathbf{u}_x \big\rangle_{\boldsymbol{\phi}} = \big[\mathbf{b}(\mathbf{u})^\top \mathbf{u}_x\big]_{\boldsymbol{\phi}}$$

In particular, for given $\mathbf{b} : \mathbb{R}^N \mapsto \mathbb{R}^N$ we interpret

$$\big\langle \mathbf{b}(\mathbf{u}), \mathbf{f}_j(\mathbf{u})_{x_j} \big\rangle_{\boldsymbol{\phi}} = \big[\big(\mathbf{b}(\mathbf{u})^\top A_j(\mathbf{u})\big)\mathbf{u}_{x_j}\big]_{\boldsymbol{\phi}}.$$

The DLM theory extends Volpert theory [81] for non-conservative products which is based on the straight path, $\boldsymbol{\phi}(s, \mathbf{u}_-, \mathbf{u}_+) = (1-s)\mathbf{u}_- + s\mathbf{u}_+$. Note that the variational statement takes into account a proper family of locally Lipschitz paths which impact the value of the non-conservative products.

2.1. **Weak solutions.** A variational solution $\overline{\mathbf{u}}$ should minimize $\mathcal{I}_{\eta_{\mathbf{c}}}(\mathbf{u})$. In particular, since $\nabla\eta_{\mathbf{c}} = \nabla\eta + \mathbf{c}$, we have

$$\mathcal{I}_\eta(\overline{\mathbf{u}}) \leqslant \mathcal{I}_{\eta_{\mathbf{c}}}(\overline{\mathbf{u}}) = \mathcal{I}_\eta(\overline{\mathbf{u}}) + \int_{t=t_1}^{t_2} \int_\Omega \Big\langle \mathbf{c}, \overline{\mathbf{u}}_t + \sum_{j=1}^d \mathbf{f}_j(\overline{\mathbf{u}})_{x_j} \Big\rangle_{\boldsymbol{\phi}} \, \mathrm{d}\mathbf{x}\, \mathrm{d}t,$$

hence

$$\int_{t=t_1}^{t_2} \int_\Omega \Big\langle \mathbf{c}, \overline{\mathbf{u}}_t + \sum_{j=1}^d \mathbf{f}_j(\overline{\mathbf{u}})_{x_j} \Big\rangle_{\boldsymbol{\phi}} \, \mathrm{d}\mathbf{x}\, \mathrm{d}t \geqslant 0 \quad \text{for all} \quad \mathbf{c} \in \mathbb{R}^N.$$

In the present case of a Jacobian coefficient matrix, the DLM product coincides with the distributional derivative of the corresponding flux, $\mathbf{f}_j(\mathbf{u})_{x_j} = \langle A_j(\mathbf{u}), \mathbf{u}_{x_j}\rangle_{\boldsymbol{\phi}}$, and is therefore independent of the path $\boldsymbol{\phi}$. Since the integrand is conservative, its integral is path independent, and we conclude

$$\begin{aligned}
&\int_{t=t_1}^{t_2} \int_\Omega \Big\{ \overline{u}^\alpha_t + \operatorname{div} \mathbf{f}^\alpha(\overline{\mathbf{u}}) \Big\} \, \mathrm{d}\mathbf{x}\, \mathrm{d}t \\
&= \int_\Omega \overline{u}^\alpha(t,\cdot)\, \mathrm{d}\mathbf{x}\Big|_{t=t_1}^{t=t_2} + \int_{t=t_1}^{t_2} \int_{\partial\Omega} \mathbf{f}^\alpha(\overline{\mathbf{u}}) \cdot \mathbf{n}\; \mathrm{d}S\, \mathrm{d}t = 0, \qquad \alpha = 1, \ldots, N.
\end{aligned}$$

Corollary 2.3. *A variational solution is a weak solution of* (1).

The "linearization" argument of $\eta_{\mathbf{c}}$ for large $|\mathbf{c}| > \|\nabla\eta(\mathbf{u})\|_{L^\infty}$ which recovers $\overline{\mathbf{u}}$ as a weak solution of the "underlying" conservation law, is reminiscent of the recovery of scalar weak solutions, u, by linearization of Kružkov's entropies (7), $\eta_c = |u - c|$ for large $c > \|u(t,\cdot)\|_{L^\infty}$.

It is clear that the closure of $\mathcal{E}$ under linear perturbations, $\{\eta_{\mathbf{c}} : \mathbf{c} \in \mathbb{R}^N\}$, is in fact equivalent to having the class of variational solutions, $\overline{\mathbf{u}}$, include weak solutions of (1). In what follows, we shall need to consider perturbations of such weak solutions. Accordingly, we let the set of admissible states, $\mathcal{W}$, include arbitrarily small ϵ-perturbations of weak solutions,

$$\begin{aligned}
&\mathcal{W}(t_1, t_2) \\
&= \left\{ \mathbf{u} + \delta\mathbf{u} \ : \ \mathbf{u} \text{ is a weak solution of (2)}; \ \left\{ \begin{array}{l} \delta\mathbf{u} \in C_0^\infty\big((t_1,t_2),\Omega\big) \\ \|\delta\mathbf{u}\|_\infty < \epsilon \end{array} \right\} \right\}.
\end{aligned} \tag{19}$$

The notion of a variational solution now takes the following form.

Variational solutions revisited. I. $(\eta, \overline{\mathbf{u}})$ is a variational solution of (1) if η belongs to the non-empty set of observables $\mathcal{E}$, and $\overline{\mathbf{u}}$ is a weak solution which is a minimizer of $\mathcal{I}_\eta(\mathbf{u})$ observed by η, such that for all perturbed $\mathbf{u} \in \mathcal{W}$ with arbitrarily small ϵ,

$$\mathcal{I}_\eta\big(\overline{\mathbf{u}}; (t_1, t_2)\big) \leqslant \mathcal{I}_\eta\big(\mathbf{u}; (t_1, t_2)\big) \text{ for all } \left\{ \begin{array}{ll} \mathbf{u}(t, \cdot) \in \mathcal{W}(t_1, t_2), & t \in (t_1, t_2) \\ \mathbf{u}(t, \cdot) = \overline{\mathbf{u}}(t, \cdot), & t \leqslant t_1. \end{array} \right\}. \tag{20}$$

The variational formulation (20) is based on a "push forward" action functional $\mathcal{I}_\eta(\mathbf{u})$. In particular, the admissible states, $\mathbf{u}(t, \cdot)$ — and therefore the variational solution $\overline{\mathbf{u}}(t, \cdot)$ — are anchored in the initial data, $\mathbf{u}(0, \cdot) = \mathbf{u}_0$, prescribed at $t_1 = 0$. Thereafter, the states $\mathbf{u}(t, \cdot),\ t > t_1 > 0$ are perturbations of weak solutions which evolve from the variational solution $\overline{\mathbf{u}}(t_1, \cdot)$.

2.2. **Derivation of entropy functions.** We compute the formal first variation in $\mathcal{I}_\eta(\mathbf{u})$ using smooth perturbations, $\|\delta\mathbf{u}\|_\infty < \epsilon$, compactly supported inside $(t_1, t_2) \times \Omega$,

$$\begin{aligned} \mathcal{I}_\eta(\mathbf{u} + \delta\mathbf{u}) - \mathcal{I}_\eta(\mathbf{u}) \approx \int\limits_{t=t_1}^{t_2} \int\limits_{\Omega} \Big\{ \Big\langle D^2\eta(\mathbf{u})\delta\mathbf{u}, \mathbf{u}_t + \sum_{j=1}^{d} A_j(\mathbf{u})\mathbf{u}_{x_j} \Big\rangle_{\boldsymbol{\phi}} \\ + \Big\langle \nabla\eta(\mathbf{u}), (\delta\mathbf{u})_t + \sum_{j=1}^{d} \big(A_j(\mathbf{u})\delta\mathbf{u}\big)_{x_j} \Big\rangle_{\boldsymbol{\phi}} \Big\} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t. \end{aligned}$$

The approximation on the right takes into account first-order perturbations in $\delta\mathbf{u}$, and ignores terms of order $\mathcal{O}(\epsilon^2)$, given that $\mathbf{u}$ has bounded amplitude in $BV \cap L^\infty$. The two terms on the right that involve time derivatives cancel each other

$$\begin{aligned} &\int\limits_{t=t_1}^{t_2} \int\limits_{\Omega} \Big\{ \big\langle D^2\eta(\mathbf{u})\delta\mathbf{u}, \mathbf{u}_t \big\rangle_{\boldsymbol{\phi}} + \big\langle \nabla\eta(\mathbf{u}), (\delta\mathbf{u})_t \big\rangle_{\boldsymbol{\phi}} \Big\} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\ &= \int\limits_{t=t_1}^{t_2} \int\limits_{\Omega} \big\langle \nabla\eta(\mathbf{u}), \delta\mathbf{u} \big\rangle_t \, \mathrm{d}\mathbf{x} \, \mathrm{d}t = \int\limits_{\Omega} \big\langle \nabla\eta(\mathbf{u}), \delta\mathbf{u} \big\rangle \, \mathrm{d}\mathbf{x} \Big|_{t=t_1}^{t=t_2} = 0. \end{aligned}$$

For the remaining spatial terms we have

$$\begin{aligned} &\sum_{j=1}^{d} \int\limits_{t=t_1}^{t_2} \int\limits_{\Omega} \Big\{ \Big\langle D^2\eta(\mathbf{u})\delta\mathbf{u}, A_j(\mathbf{u})\mathbf{u}_{x_j} \Big\rangle_{\boldsymbol{\phi}} + \Big\langle \nabla\eta(\mathbf{u}), (A_j(\mathbf{u})\delta\mathbf{u})_{x_j} \Big\rangle_{\boldsymbol{\phi}} \Big\} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\ &= \sum_{j=1}^{d} \int\limits_{t=t_1}^{t_2} \int\limits_{\Omega} \Big\{ \Big[(\delta\mathbf{u})^\top D^2\eta(\mathbf{u}) A_j(\mathbf{u})\mathbf{u}_{x_j} \Big]_{\boldsymbol{\phi}} - \Big[(\delta\mathbf{u})^\top A_j^\top(\mathbf{u}) D^2\eta(\mathbf{u})\mathbf{u}_{x_j} \Big]_{\boldsymbol{\phi}} \Big\} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t. \end{aligned} \tag{21}$$

This follows from the following key feature of the DLM-Volpert theory: whenever a non-conservative product $\big[\mathbf{b}(\mathbf{u})^\top \mathbf{u}_x\big]_{\boldsymbol{\phi}}$ forms a "perfect derivative", it coincides with the standard distributional derivative and is independent of the path $\boldsymbol{\phi}$, [17, Proposition 1.5],[81, §13.2]; in particular,

$$\big\langle \mathbf{b}(\mathbf{u})_x, \mathbf{u} \big\rangle_{\boldsymbol{\phi}} + \big\langle \mathbf{b}(\mathbf{u}), \mathbf{u}_x \big\rangle_{\boldsymbol{\phi}} = \Big[\Big(\mathbf{u}^\top \frac{\partial \mathbf{b}(\mathbf{u})}{\partial \mathbf{u}} + \mathbf{b}(\mathbf{u})^\top \Big) \mathbf{u}_x \Big]_{\boldsymbol{\phi}} = \big\langle \mathbf{b}(\mathbf{u}), \mathbf{u} \big\rangle_x,$$

is independent of the path $\boldsymbol{\phi}$, which justifies the "integration by parts" on the second integrand in (21).

Since we may choose arbitrary smooth, compactly supported perturbations $\delta\mathbf{u}$, the requirement that $(\delta_{\mathbf{u}}\mathcal{I}_\eta)(\overline{\mathbf{u}}) = 0$ implies

$$\big[D^2\eta(\overline{\mathbf{u}})A_j(\overline{\mathbf{u}})\overline{\mathbf{u}}_{x_j}\big]_{\boldsymbol{\phi}} = \big[A_j^\top(\overline{\mathbf{u}})D^2\eta(\overline{\mathbf{u}})\overline{\mathbf{u}}_{x_j}\big]_{\boldsymbol{\phi}}, \tag{22}$$

and since this is intended to hold for arbitrary $\overline{\mathbf{u}}_{x_j}$ and independently of the choice of path, we are led to

$$D^2\eta(\overline{\mathbf{u}})A_j(\overline{\mathbf{u}}) = A_j^\top(\overline{\mathbf{u}})D^2\eta(\overline{\mathbf{u}}), \quad j = 1, \ldots, d.$$

This is precisely the symmetrizability condition (6) restricted to the variational solution $\overline{\mathbf{u}}$. At the formal level, if the variational principle is required to hold for a sufficiently rich class of weak solutions, then the stationarity condition forces the pointwise symmetrization relation (6).

In this manner the notion of an entropy function is deduced from the variational principle (17) or (20): $\eta \in \mathcal{E}$ is an admissible observable if its Hessian symmetrizes the Jacobians, (6), that is, if η is an entropy function. This answers Friedrichs' question in the sense that the notion of entropy is derived as a stationary condition for the variational functional $\mathcal{I}_\eta$.

Remark 2.4 (Conservative form). The derivation of entropy symmetrization requires the conservative form of the equations. To highlight this point, we appeal to the equations in their quasi-linear form (3). Assuming the C^1-smooth setting, the first variation of the corresponding functional

$$\mathcal{I}_\eta = \int\limits_{t=t_1}^{t_2}\int\limits_{\Omega} \sum_{\alpha,\beta=1}^{N} \frac{\partial\eta(\mathbf{u})}{\partial u^\alpha} \sum_{j=0}^{d} \big(A_j(\mathbf{u})\big)_{\alpha\beta} \frac{\partial u^\beta}{\partial x_j}\, \mathrm{d}\mathbf{x}\,\mathrm{d}t,$$

(where x_0 is identified as the time variable), reads for $\gamma = 1, \ldots, N$, e.g., [32, §37.4]

$$\begin{aligned}
&\frac{\partial\mathcal{I}_\eta}{\partial u^\gamma} - \sum_{j=0}^{d}\frac{\partial}{\partial x_j}\frac{\partial\mathcal{I}_\eta}{\partial u^\gamma_{x_j}} \\
&= \int\limits_{t=t_1}^{t_2}\int\limits_{\Omega}\sum_{\alpha,\beta=1}^{N}\sum_{j=0}^{d}\left(\frac{\partial^2\eta(\mathbf{u})}{\partial u^\alpha\partial u^\gamma}\big(A_j(\mathbf{u})\big)_{\alpha\beta}\frac{\partial u^\beta}{\partial x_j} + \frac{\partial\eta(\mathbf{u})}{\partial u^\alpha}\frac{\partial\big(A_j(\mathbf{u})\big)_{\alpha\beta}}{\partial u^\gamma}\frac{\partial u^\beta}{\partial x_j}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad - \int\limits_{t=t_1}^{t_2}\int\limits_{\Omega}\sum_{\alpha=1}^{N}\sum_{j=0}^{d}\frac{\partial}{\partial x_j}\left(\frac{\partial\eta(\mathbf{u})}{\partial u^\alpha}\big(A_j(\mathbf{u})\big)_{\alpha\gamma}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \int\limits_{t=t_1}^{t_2}\int\limits_{\Omega}\sum_{\alpha,\beta=1}^{N}\sum_{j=0}^{d}\left(\frac{\partial^2\eta(\mathbf{u})}{\partial u^\alpha\partial u^\gamma}\big(A_j(\mathbf{u})\big)_{\alpha\beta}\frac{\partial u^\beta}{\partial x_j} - \frac{\partial^2\eta(\mathbf{u})}{\partial u^\alpha\partial u^\beta}\big(A_j(\mathbf{u})\big)_{\alpha\gamma}\frac{\partial u^\beta}{\partial x_j}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad + \int\limits_{t=t_1}^{t_2}\int\limits_{\Omega}\sum_{\alpha,\beta=1}^{N}\sum_{j=0}^{d}\left(\frac{\partial\eta(\mathbf{u})}{\partial u^\alpha}\frac{\partial\big(A_j(\mathbf{u})\big)_{\alpha\beta}}{\partial u^\gamma}\frac{\partial u^\beta}{\partial x_j} - \frac{\partial\eta(\mathbf{u})}{\partial u^\alpha}\frac{\partial\big(A_j(\mathbf{u})\big)_{\alpha\gamma}}{\partial u^\beta}\frac{\partial u^\beta}{\partial x_j}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}$$

The last integrand on the right vanishes because of the conservative form of the equations

$$\frac{\partial\big(A_j(\mathbf{u})\big)_{\alpha\beta}}{\partial u^\gamma} = \frac{\partial f_j^\alpha}{\partial u^\beta\partial u^\gamma} = \frac{\partial\big(A_j(\mathbf{u})\big)_{\alpha\gamma}}{\partial u^\beta},$$

hence the Euler equations imply that $\overline{\mathbf{u}}$ is a stationary point for $\mathcal{I}_\eta(\mathbf{u})$ provided the preceding integral vanishes as well, namely

$$\delta\mathcal{I}_\eta(\overline{\mathbf{u}}) = \sum_{j=0}^{d} \int_{t=t_1}^{t_2} \int_{\Omega} \Big(\big[D^2\eta(\overline{\mathbf{u}})A_j(\overline{\mathbf{u}}) - A_j^\top(\overline{\mathbf{u}})D^2\eta(\overline{\mathbf{u}})\big]\, \overline{\mathbf{u}}_{x_j} \Big)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t = 0.$$

The requirement $\delta\mathcal{I}(\overline{\mathbf{u}}) = 0$ for arbitrary $\overline{\mathbf{u}}_{x_j}$ recovers the symmetrization (6).

Remark 2.5 (**Entropic vs. non-entropic systems).** The notion of variational solution does not treat the set of observables as an "extension" which augments the hyperbolic system of conservation laws, but as an intrinsic part of the solution. Since observables coincide with entropy functions, the set of observables $\mathcal{E}$ is restricted. Therefore, we make a distinction between two classes of systems of equations:

Entropic systems which admit non-empty set $\mathcal{E}$ — this is the "interesting" class of conservation laws of Godunov which admit (at least one) entropy function. This is the class of systems discussed in this paper. In case there is more than one entropy function, we can fix one preferred $\eta \in \mathcal{E}$ and we refer to $\overline{\mathbf{u}}$ as a variational solution observed by that η. At the same time, we can discuss a variational solution observed by all $\eta \in \mathcal{E}$. Then there is the other class of systems of conservation laws, at least when $N \geqslant 3$, which admit no entropy extension, i.e., where $\mathcal{E}$ is empty, as in (8). This class of "non-entropic" systems do not admit variational solutions. The study of variational solutions for such systems requires a different setting.

We close by noting the following duality. The closure of the set of observables with respect affine perturbations, $\eta_{\mathbf{c}} \in \mathcal{E}$, enforces variational solutions to be weak solutions, $\overline{\mathbf{u}} \in \mathcal{W}$, while letting the set of admissible states $\mathcal{W}$ include local perturbations of weak solutions, $\mathbf{u} + \delta\mathbf{u} \in \mathcal{W}$, enforces the observables to be entropy functions, $\eta \in \mathcal{E}$. In this context, the class of perturbations considered in (19), which consists of all $\{\delta\mathbf{u}\}$ in the L^∞ ϵ-ball, is a "rich" class. It should be possible to consider a restricted class of perturbations and consequently, to further restrict the set of perturbed weak solutions $\mathcal{W}(t_1, t_2)$ in (19). A key question of interest is *to identify a minimal set of perturbed weak solutions which will enforce* (22).

3. **The entropy inequality.** Fix $\eta \in \mathcal{E}$. Since its Hessian symmetrizes the Jacobians, $A_j(\mathbf{u})$, it is a part of an entropy-entropy flux pair, $(\eta, \mathbf{q}^\eta)$, such that (4) holds, and the variational functional, $\mathcal{I}_\eta(\mathbf{u})$ now amounts to

$$\begin{aligned}\mathcal{I}_\eta\big(\mathbf{u};(t_1,t_2)\big) &= \int_{t_1}^{t_2} \int_{\Omega} \big\{\eta(\mathbf{u})_t + \nabla_{\mathbf{x}} \cdot \mathbf{q}^\eta(\mathbf{u})\big\}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \\ &= \int_{\Omega} \eta\big(\mathbf{u}(t,\mathbf{x})\big)\, \mathrm{d}\mathbf{x}\Big|_{t=t_1}^{t=t_2} + \int_{t=t_1}^{t_2} \int_{\partial\Omega} \mathbf{q}^\eta\big(\mathbf{u}(t,\mathbf{x})\big) \cdot \mathbf{n}\ \mathrm{d}S\, \mathrm{d}t.\end{aligned} \tag{23}$$

The notion of variational solution in (20) now reads as follows.

Variational solutions revisited. II. Given an entropy function $\eta \in \mathcal{E}$, then $(\eta, \overline{\mathbf{u}})$ is a variational solution of (1) if $\overline{\mathbf{u}}$ is a weak solution which is a minimizer of $\mathcal{I}_\eta(\mathbf{u})$

among all weak solutions $\mathbf{u}$ which agree with $\overline{\mathbf{u}}$ at $t = t_1$,

$$\begin{aligned}\int_{\Omega} \eta\big(\overline{\mathbf{u}}(t,\mathbf{x})\big)\,\mathrm{d}\mathbf{x}\Big|_{t=t_1}^{t=t_2} + \int_{t=t_1}^{t_2}\int_{\partial\Omega} \mathbf{q}^{\eta}\big(\overline{\mathbf{u}}(t,\mathbf{x})\big)\cdot\mathbf{n}\,\mathrm{d}S\,\mathrm{d}t \\ \leqslant \int_{\Omega} \eta\big((\mathbf{u}+\delta\mathbf{u})(t,\mathbf{x})\big)\,\mathrm{d}\mathbf{x}\Big|_{t=t_1}^{t=t_2} + \int_{t=t_1}^{t_2}\int_{\partial\Omega} \mathbf{q}^{\eta}\big((\mathbf{u}+\delta\mathbf{u})(t,\mathbf{x})\big)\cdot\mathbf{n}\,\mathrm{d}S\,\mathrm{d}t.\end{aligned} \tag{24}$$

Note that (24) does *not* recover the entropy inequality (5), which in its weak form reads

$$\int_{\Omega} \eta\big(\overline{\mathbf{u}}(t,\mathbf{x})\big)\,\mathrm{d}\mathbf{x}\Big|_{t=t_1}^{t=t_2} + \int_{t=t_1}^{t_2}\int_{\partial\Omega} \mathbf{q}^{\eta}\big(\overline{\mathbf{u}}(t,\mathbf{x})\big)\cdot\mathbf{n}\,\mathrm{d}S\,\mathrm{d}t \leqslant 0. \tag{25}$$

Indeed, the settings for *entropy solutions* satisfying (25), and for variational solutions satisfying (24), are different. Entropy solutions are realized by vanishing viscosity limits, $\mathbf{u} = \lim_{\epsilon\to 0} \mathbf{u}^{\epsilon}$, where

$$\eta(\mathbf{u}^{\epsilon})_t + \nabla_{\mathbf{x}}\cdot\mathbf{q}^{\eta}(\mathbf{u}^{\epsilon}) \leqslant \epsilon\Delta_{\mathbf{x}}\eta(\mathbf{u}^{\epsilon}), \qquad \mathbf{u}^{\epsilon}_t + \sum_{j=1}^{d}\mathbf{f}_j(\mathbf{u}^{\epsilon})_{x_j} = \epsilon\Delta_{\mathbf{x}}\mathbf{u}^{\epsilon}, \qquad \epsilon > 0.$$

Viscosity perturbations impose an instantaneous decrease of entropy. A celebrated result of Lax, [56, 57], shows that this identifies physically relevant shock discontinuities, encoded by nontrivial defect measure $\lim_{\epsilon\downarrow 0}\big\{-\epsilon|\nabla_{\mathbf{x}}\eta(\mathbf{u}^{\epsilon}(t,\cdot))|^2\big\} < 0$. The notion of variational solutions replaces viscosity perturbations, $\mathbf{u}\mapsto\mathbf{u}^{\epsilon}$, by local smooth perturbations, $\mathbf{u}\mapsto\mathbf{u}+\delta\mathbf{u}\in\mathcal{W}$, which, a priori, need not decrease the entropy production. See Example 5.1 below.

Remark 3.1. It would be interesting to develop an alternative theory of variational solutions based on the class of Lipschitz smooth perturbations, $\mathcal{W} = \{\mathbf{u}_{\boldsymbol{\phi}}\}$, where the non-conservative product $[\mathbf{b}(\mathbf{u})^{\top}\mathbf{u}_x]_{\boldsymbol{\phi}}$ is replaced by the limit of $\mathbf{b}(\mathbf{u}_{\boldsymbol{\phi}})^{\top}\dfrac{\mathrm{d}\mathbf{u}_{\boldsymbol{\phi}}}{\mathrm{d}s}$, [17, §4].

The entropy inequality, $\eta(\mathbf{u})_t + \nabla_{\mathbf{x}}\cdot\mathbf{q}^{\eta}(\mathbf{u}) \leqslant 0$, was set as a selection principle to identify a unique, physically relevant solution, [52, Definition 2]. This selection principle completely settled the question of uniqueness in the scalar case, being endowed with the "rich" family of *all* convex entropy functions, [52]. But most systems are not endowed with rich family of entropies, and those in mathematical physics are identified with one preferred entropy, usually the one driven by thermodynamic considerations. Therefore, one is interested to address the uniqueness question in the setting "observed" by a single preferred entropy function. As noted in Remark 2.5, this is the preferred setting of variational solutions which emphasizes a single entropy function (or at least a restricted set $\mathcal{E}$ of entropies). We mention in this context two canonical results. Panov [69] proved the uniqueness of quadratic entropy solutions for one-dimensional scalar conservation laws, $u_t + f(u)_x = 0$, with convex flux $f(u)$. See also [63, 51]. For arbitrary systems of entropic conservation laws, the use of a single *relative entropy* function, initiated in [14], secures uniqueness within the class of strong solutions[7]. However, a selection principle based on a single entropy inequality fails in the general setting of systems of conservation laws in $N \geqslant 2$ dimensions, see §5.1 below.

4. **Maximum entropy production.** The possible lack of uniqueness in the entropy-based selection principle based on a preferred single entropy inequality (5), motivated Dafermos as early as 1973 [13], to formulate an *entropy rate admissibility criterion*. Given a strictly convex entropy η, it seeks a solution $\overline{\mathbf{u}}$ which maximizes the entropy dissipation *rate* among all other weak solutions

$$\frac{\mathrm{d}_+}{\mathrm{d}t}\int \eta(\overline{\mathbf{u}}(t,\mathbf{x}))\,\mathrm{d}\mathbf{x} \leqslant \frac{\mathrm{d}_+}{\mathrm{d}t}\int \eta(\mathbf{u}(t,\mathbf{x}))\,\mathrm{d}\mathbf{x} \ : \ \begin{cases} \mathbf{u} \text{ is a weak solution} \\ \mathbf{u}(\tau,\cdot) = \overline{\mathbf{u}}(\tau,\cdot), \quad \tau \leqslant t. \end{cases} \tag{26}$$

That is, there is no weak solution $\mathbf{u}(\tau,\cdot)$ which coincides with $\overline{\mathbf{u}}(\tau,\cdot)$ for $\tau < t$, and for which the *reverse* of (26) holds.

This can be interpreted in the general context of maximization of the entropy production, the so-called MaxEnt principle, which was developed in different contexts during the second half of the 20th century, with contributions (and sometimes controversies) from I. Prigogine (1947), J. M. Ziman (1956), E. T. Jaynes (1957), H. Ziegler (1963), R.C. Dewar [2007] and many others listed in, e.g., [42, 67][8]. Dafermos proved that, in the context of hyperbolic conservation laws, his MaxEnt criterion singles out the unique entropy solution in scalar equations and 1D p-system [13] and selects the admissible solution of 1D Riemann problem, at least with small initial variation [74, 15], and it was utilized in a host of different applications, e.g., [44, 48].

4.1. **A local version of MaxEnt.** We appeal to the variational formulation (24). Given that the observable η is an entropy function, (24) identifies $\overline{\mathbf{u}}$ as a variational solution which maximizes the local entropy production *rate*, in the sense that $\overline{\mathbf{u}}$ is a minimizer among all weak solutions, $\mathbf{u}$,

$$\begin{aligned} &\frac{\mathrm{d}_+}{\mathrm{d}t}\int\limits_{\Omega} \eta(\overline{\mathbf{u}}(t,\mathbf{x}))\,\mathrm{d}\mathbf{x} + \int\limits_{\partial\Omega} \mathbf{q}^{\eta}(\overline{\mathbf{u}}(t,\mathbf{x}))\cdot\mathbf{n}\ \mathrm{d}S \\ &\leqslant \frac{\mathrm{d}_+}{\mathrm{d}t}\int\limits_{\Omega} \eta(\mathbf{u}(t,\mathbf{x}))\,\mathrm{d}\mathbf{x} + \int\limits_{\partial\Omega} \mathbf{q}^{\eta}(\mathbf{u}(t,\mathbf{x}))\cdot\mathbf{n}\ \mathrm{d}S \ : \ \begin{cases} \mathbf{u} \text{ is a weak solution} \\ \mathbf{u}(\tau,\cdot) = \overline{\mathbf{u}}(\tau,\cdot), \quad \tau \leqslant t. \end{cases} \end{aligned} \tag{27}$$

This is a *local* version of Dafermos' MaxEnt criterion (26), localized to arbitrary $\Omega \subset \mathbb{R}^d$. It is not assumed, but is deduced directly from the variational principle: dividing (24) by $t_2 - t_1$ and letting $t_2 \to t_1 = t$ while noting that $\delta\mathbf{u}$ vanishes as well.

At this stage we can summarize the three main consequences of our variational principle outlined in Definition 2.1, namely — variational solutions are weak solution; stationary observable must be an entropy function; and if η is an entropy function, the variational principle implies local MaxEnt principle.

[7] As noted by Dafermos [16, §5] "It is remarkable that a single entropy inequality, with convex entropy, manages to weed out all but one solution of the initial value problem, so long as a classical solution exists."

[8] As noted in [37], Dafermos' MaxEnt criterion relates to Ziegler's principle for closed thermodynamic systems rather than Prigogine's MinEnt principle which applies to open thermodynamic systems

5. In search of a uniqueness selection principle.

5.1. The entropy inequality as a selection principle. There is a large body of recent work that addressed the (non-)uniqueness question of entropy solutions. The canonical system addressed in this large body of work is the isentropic Euler equations, which consists of $N = d + 1$ equations for $(\rho, \rho\mathsf{v}) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$\mathbf{u} = \begin{bmatrix} \rho \\ \rho\mathsf{v} \end{bmatrix}, \quad \mathbf{f}_j(\mathbf{u}) = \begin{bmatrix} \rho\mathsf{v}_j \\ \rho\mathsf{v}_j\mathsf{v} + p\delta_{ij} \end{bmatrix}, \quad j = 1, \ldots, d, \tag{28}$$

with pressure $p = p(\rho)$. The γ-law, $p(\rho) = \rho^\gamma$, corresponds to Euler system (15) with constant specific entropy S. Here, the energy $\eta(\rho, \rho\mathsf{v}) := \dfrac{1}{2}\rho|\mathsf{v}|^2 + \rho \displaystyle\int^{\rho} \frac{p(r)}{r^2}\,\mathrm{d}r$ is serving as the entropy function.

The relative entropy-based uniqueness within the class of strong rarefaction-based solutions was proved in [6, 23]. Following the initial breakthrough of De Lellis and Székelyhidi in [18] which used convex integration to construct non-unique "wild" solutions for bounded data, a series of results of non-uniqueness with improved regularity followed with regular density [7], compactly supported data [1], 2D Lipschitz and smooth initial data [8, 11], and even non-unique entropy conservative Hölder solutions [35]. General non-uniqueness for shock-based solutions was proved in [9, 10, 66] and was extended to the full Euler equations in [49]. These works make clear that the selection principle based on a single entropy inequality to single out a unique solution among many weak solutions for the multidimensional isentropic equations (28) does not suffice to address the uniqueness problem.

Lax commented on this situation in his Gibbs lecture [61] "*Just because we cannot prove that compressible flows with prescribed initial values exist doesn't mean that we cannot compute them.*" Numerical evidence for non-uniqueness of entropic solutions of Euler equations was further investigated in [26].

Example 5.1 (Increase of entropy). Like most of the theoretical aspects in this field, the entropy inequality (5) was motivated by entropic solutions of the compressible Euler equations (15), which are realized as vanishing viscosity limits of Navier-Stokes equations (NSe) (here ∇_S denotes the symmetric gradient $\nabla_S \cdot = \frac{1}{2}\big(\nabla \cdot + (\nabla \cdot)^\top\big)$)

$$\mathbf{u}^{\text{Euler}}_t + \sum_{j=1}^{d} \mathbf{f}^{\text{Euler}}_j(\mathbf{u})_{x_j} = \begin{bmatrix} 0 \\ \operatorname{div} \mathbb{T} \\ \nabla \cdot (\mathbb{T}\mathsf{v} + \kappa\nabla\theta) \end{bmatrix}, \quad \mathbb{T} := 2\mu\nabla_S\mathsf{v} + \lambda(\nabla \cdot \mathsf{v})\mathbb{I}. \tag{29}$$

The Navier-Stokes equations (NSe) are *the* canonical "vanishing viscosity" perturbation of Euler equations (15). With zero heat conduction, $\kappa = 0$, they yield the entropy inequality for the convex entropy[9] $\eta(\mathbf{u}^{\text{Euler}}) = -\rho S$, uniformly as $\mu, \lambda > 0$,

$$(-\rho S)_t + \nabla_{\mathbf{x}} \cdot (-\rho\mathsf{v}S) < 0, \qquad S = \ln(p\rho^{-\gamma})$$

[9]The system of Euler equations is equipped with the family of entropy functions (16). When extended to the system of Navier-Stokes equations with heat conduction, (29), only the "physical" entropy, $\eta(\mathbf{u}^{\text{Euler}}) = -\rho S$, survives as the one which puts the additional viscosity *and* heat fluxes into a symmetric negative-definite form, [46], and the entropy inequality, $(-\rho S)_t + \nabla_{\mathbf{x}} \cdot (-\rho\mathsf{v}S + \kappa\nabla \ln\theta) < 0$, follows.

However, the decrease of entropy is not shared by all well-posed perturbations of Euler equations. Consider the Euler alignment system, driven by a radial communication kernel $0 < k \leqslant 1$,

$$\mathbf{u}_t^{\text{Euler}} + \sum_{j=1}^{d} \mathbf{f}_j^{\text{Euler}}(\mathbf{u})_{x_j} = \begin{bmatrix} 0 \\ \tau \displaystyle\int k(|\mathbf{x}-\mathbf{x}'|)\big(\mathsf{v}'-\mathsf{v}\big)\rho\rho'\,\mathrm{d}\mathbf{x}' \\ -\tau \displaystyle\int k(|\mathbf{x}-\mathbf{x}'|)\big(2E-\mathsf{v}'\cdot\mathsf{v}\big)\rho\rho'\,\mathrm{d}\mathbf{x}' \end{bmatrix} \quad \mathsf{v}'=\mathsf{v}(t,\mathbf{x}'). \tag{30}$$

The system admits global weak solutions [12]. These are in fact strong solutions for sub-critical initial data: it was shown in [80] that Euler alignment, viewed as a perturbation of compressible Euler system (15), yields the *reverse* inequality uniformly in $\tau > 0$,

$$(-\rho S)_t + \nabla_{\mathbf{x}} \cdot (-\rho \mathsf{v} S) > 0.$$

It reflects the increasing order, or emergence in the Euler alignment system (30), and it demonstrates that imposing the "direction" of an entropy inequality (5) is connected with the class of perturbations applied to the underlying systems of conservation laws.

5.2. **The MaxEnt condition as a selection principle.** Hsiao [45] found a surprisingly simple counterexample of 1D Riemann problem for the compressible Euler equations, where Dafermos' MaxEnt principle fails to single out a unique solution within the class of piecewise smooth solutions, when $\gamma > 5/3$. It turns out that Dafermos' criterion fails as a uniqueness selection principle for a certain class of initial data for the multidimensional isentropic Euler equations [9, 8], that is, there exist wild solutions with better entropy dissipation rate when compared with the self-similar solution of the 1D Riemann problem extended to two dimensions. In fact, even the local maximal dissipation criterion (27) fails as a uniqueness criterion [65].

Alternative approaches to Dafermos' MaxEnt principle were offered by the least action principle in [33, 34], the notion of energy-variational solutions [21], as well as measure-valued solutions realized by low Mach limits or vanishing viscosity limits [30, 31], but they fail to secure uniqueness.

The MaxEnt principle of Dafermos was also examined in the context of the larger class of *measure valued* solutions [20] for (1), where various variational-based selection criteria were examined to single out a unique measure-valued (m.v.) solution, primarily for the isentropic Euler equations (28). While the MaxEnt fails for m.v. solutions, [25], it was shown in [22] that a local MaxEnt selection principle identifies m.v. solutions of (28), thus proving a local version of DiPerna's conjecture [20]. An alternative two-step selection criterion was studied in [24], improving the earlier multi-step selection criterion in [2]. A different criterion, the MaxVar criterion for m.v. solutions, was studied in [50]; this remains a work in progress.

5.3. **A variational formulation as a selection principle.** Markfelder [65] proved that the local MaxEnt (27) fails for the isentropic equations (28) with $\gamma = 2$. The variational formulation (24) is in fact more restrictive than the local MaxEnt criterion (27), in the sense that it requires minimization over local perturbations of weak solutions. Its global version reads

$$\int \eta\big(\overline{\mathbf{u}}(t,\mathbf{x})\big)\,\mathrm{d}\mathbf{x}\Big|_{t=t_1}^{t=t_2} \leqslant \int \eta\big((\mathbf{u}+\delta\mathbf{u})(t,\mathbf{x})\big)\,\mathrm{d}\mathbf{x}\Big|_{t=t_1}^{t=t_2} \ \text{for all} \ \left\{ \begin{array}{l} \mathbf{u}+\delta\mathbf{u} \in \mathcal{W}(t_1,t_2) \\ \mathbf{u}(t,\cdot)=\overline{\mathbf{u}}(t,\cdot). \end{array} \right\}.$$

This may offer an alternative to the entropy admissibility criteria which may not always serve as appropriate selection principles, [34, Remark 9.5]. It remains an open question to identify a suitable space $\mathcal{W}$ which would provide a fully rigorous treatment for the questions of a single entropy-based selection criterion for existence and uniqueness of variational solutions,

Appendix A. **Conservation laws with homogeneous fluxes.** The compressible Euler equations (15) have the distinctive property of spatial fluxes which are homogeneous, $\mathbf{f}_j^{\text{Euler}}(\lambda\mathbf{u}) = \lambda^\beta \mathbf{f}_j^{\text{Euler}}(\mathbf{u})$ of degree $\beta = 1$. François Golse noted[10] that this is a direct consequence of the fact that Euler equations can be formally recovered from a limiting BGK model with Maxwellian $M_{(\rho,\mathsf{v},\theta)}(\boldsymbol{\xi}) = \dfrac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|\boldsymbol{\xi}-\mathsf{v}|^2}{2\theta}}$ which is homogeneous of degree one, $M_{\lambda(\rho,\mathsf{v},\theta)}(\boldsymbol{\xi}) = \lambda M_{(\rho,\mathsf{v},\theta)}(\boldsymbol{\xi})$.

The purpose of this short appendix is to note that the homogeneous structure of Euler equations can be put in a *symmetric form*, [75]. We think that it could be further exploited in the variational formulation of Euler equations.

To this end we use the class of convex entropy functions (corresponding to $h(S) = e^{S/(\alpha+\gamma)}$)

$$\eta(\mathbf{u}^{\text{Euler}}) = -(p\rho^\alpha)^{\frac{1}{\alpha+\gamma}}, \qquad \alpha \geqslant 0 \tag{31}$$

which are homogeneous of degree $\beta_\alpha = \dfrac{\alpha+1}{\alpha+\gamma}$. Expressed in terms of the entropy variables, $\mathbf{v} = \nabla\eta(\mathbf{u}^{\text{Euler}})$, the Euler equations admit the symmetric form

$$\mathbf{u}^{\text{Euler}}(\mathbf{v})_t + \sum_{j=1}^{d} \mathbf{f}_j^{\text{Euler}}(\mathbf{u}(\mathbf{v}))_{x_j} = \left(A_0^S\right)^{-1}\mathbf{v}_t + \sum_{j=1}^{d} A_j^S \mathbf{v}_{x_j} = 0,$$

with symmetric Jacobians, (14), A_j^S, $j = 0, 1, \ldots, d$. The temporal and spatial fluxes, $\mathbf{u}^{\text{Euler}}(\mathbf{v})$ and $\mathbf{f}_j^{\text{Euler}}(\mathbf{u}(\mathbf{v}))$, are homogeneous of degree $\dfrac{1}{\beta_\alpha - 1} = \dfrac{\alpha+\gamma}{1-\gamma} < 0$. Since the fluxes are homogeneous of degree $\dfrac{1}{\beta_\alpha - 1}$, Euler's identity enables us to rewrite (31) in the symmetric conservative form

$$\left(\left(A_0^S\right)^{-1}\mathbf{v}\right)_t + \sum_{j=1}^{d} \left(A_j^S \mathbf{v}\right)_{x_j} = 0,$$

The homogeneity of Euler equations implies that if $\mathbf{v}(t,\cdot)$ is an entropy-variable solution of Euler equations then any scalar multiple of $\mathbf{v}$ is also a solution of Euler equations. What other systems share this property?

The system of MHD equations, a natural candidate, does not share this property because of the dependence of its fluxes on dimensional parameter of magnetic permeability μ^*, [75, §5]. Another such system is the ultra-relativistic Euler system which expresses the conservation of momentum $\mathbf{m} = p\mathsf{v}$ and energy $E = 3p + p|\mathsf{v}|^2$,

[10]Private communication.

e.g., [53, 54]

$$(4m_k\sqrt{1+|\mathsf{v}|^2})_t+\sum_{j=1}^d\big(4m_k\mathsf{v}_j+p\delta_{jk}\big)_{x_j}=0,\qquad k=1,\ldots,d$$

$$E_t+\sum_{j=1}^d\big(4m_j\sqrt{1+|\mathsf{v}|^2}\big)_{x_j}=0$$

and we observe that both, the temporal and spatial fluxes are homogeneous of degree 1 in $(p,\mathbf{m})$ (since v is homogeneous of degree 0 in (m,E)).

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